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On some results of the cohomology of extra special p -groups

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Extra special p -groups are central extensions of \mathbb{Z}/p by elementary abelian p -groups. These groups occupy a distinctive in the cohomology and representation theories of finite groups. Quillen decided mod 2 cohomology of the extra special 2-groups [Q]. However the corresponding calculation for odd p is still unknown. Tezuka-Yagita studied the varieties defined from its mod p cohomology [T-Y]. Extending these results, Benson-Carlson decided the mod p cohomology modulo Jacobson radical [B-C]. The radical parts seem very difficult. For the group of the order p^3 , Lewis decided the integral cohomology and Leary wrote down the mod p cohomology completely [Lw], [L2]. Minh computed the mod 3 cohomology of the group with the order 3^5 and of the exponent 3^2 [M].

One of main results of this paper, is to give the additive structure of the mod p cohomology of the group with the order p^5 and its exponent p (Theorem 8.25). The another results are existence of groups and their modules which period is exactly $2p^n$ for each n .

§1. Extra special p -group.

An extra special p -group G is a group such that its center is \mathbb{Z}/p and there is the central extension

$$(1.1) \quad 1 \longrightarrow \mathbb{Z}/p \longrightarrow G \xrightarrow{\pi} V \longrightarrow 1 \quad \text{where } V = \mathbb{F}_p^n \mathbb{Z}/p.$$

Such group is isomorphic to the n -th central product $E \dots E = E_n$ or $E_{n-1}M$ where E (resp. M) is the non abelian group of the order p^3 and exponent p (resp. p^2).

Hence we can explicitly write

$$(1.2) \quad E_n = \langle a_1, \dots, a_{2n}, c \mid [a_{2i-1}, a_{2i}] = c, c \in \text{Center} \\ [a_i, a_j] = 1 \text{ for } i < j, (i, j) = (2k-1, 2k) \\ a_k^p = c^p = 1 \quad \rangle.$$

The group $E_{n-1}M$ is written similarly except for $a_{2n}^p = c$.

Let us write by $x_i \in H^1(V) = \text{Hom}(V, \mathbb{Z}/p)$ the dual of $\pi_1(a_i)$ and write $y_i = \beta x_i$.

Then the cohomology of V is

$$H^*(V) \cong S_{2n} \otimes \Lambda_{2n} \quad \text{with } S_{2n} = \mathbb{Z}/p[y_1, \dots, y_{2n}] \text{ and } \Lambda_{2n} = \bigwedge(x_1, \dots, x_{2n}).$$

Proposition 1.3. The extension (1.1) represents element in $H^2(V)$

$$f = \sum^n x_{2i-1} x_{2i} \quad (\text{resp. } \sum^n x_{2i-1} x_{2i} + y_{2n}) \quad \text{for } G = E_n \text{ (resp. } E_{n-1}M).$$

We consider spectral sequence induced from (1.1)

$$(1.4) \quad E_2^{*,*} = H^*(V; H^*(\mathbb{Z}/p)) \\ \cong S_{2n} \otimes \Lambda_{2n} \otimes \mathbb{Z}/p[u] \otimes \Lambda(\mathbb{Z}) \implies H^*(G)$$

with $\beta z = u$. From Proposition 1.3, we know

$$(1.5) \quad d_2 z = f.$$

By transgression theorem,

$$(1.6) \quad d_3 u = \beta d_2 z = \sum y_{2i-1} x_{2i} - y_{2i} x_{2i-1}$$

$$(1.7) \quad d_{2p}^{s+1} u^{p^s} = \beta^{p^{s-1}} \beta d_3 u = \sum y_{2i-1} \beta^{p^s} x_{2i} - y_{2i} \beta^{p^s} x_{2i-1}.$$

By Kudo's transgression theorem

$$(1.8) \quad d_{2p}^{s(p-1)+1} (u^{p^{s(p-1)}} \otimes d_{2p}^{s+1} u^{p^s}) = \beta \beta^{p^s} d_{2p}^{s+1} u^{p^s} \\ = \sum y_{2i-1} \beta^{p^{s+1}} y_{2i} - y_{2i} \beta^{p^{s+1}} y_{2i-1}.$$

Let us write (1.6) = $z_n(1)$, (1.7) = $z_n(s+1)$, (1.8) = $w_n(s+1)$. Hence $E_\infty^{*,*}$ is a quotient of

$$(1.9) \quad E = S_{2n} \otimes \Lambda_{2n} / (f, z_n(1), \dots, z_n(n), w_n(1), \dots, w_n(n))$$

We also know that $u^{p^{n+1}}$ is a permanent cycle because which represents p^{n+1} -th Chern class of induced representation from a maximal elementary abelian p -group. Write by u' a corresponding element in $H^*(G)$. Then $H^*(G)$ is a

$E \otimes \mathbb{Z}/p[u']$ -module. From Benson-Carlson [B-C].

$$H^*(G)/J = E \mathbb{Z}_p[u']/J \quad \text{for the Jacobson radical } J.$$

Note the regularity of the sequence $w_1(1), \dots, w_n(n)$ in S_{2n} is shown in Tezuka-Yagita [T-Y].

§2. \widetilde{G} ; the central product of G and S^1 .

The spectral sequence (1.4) is very complicated even for E (see [K-S-T-Y] (5.11)). Hence we consider another arguments which are used by Kropholler, Leary, Huebshmann and Moselle. Embed $\langle c \rangle \cong \mathbb{Z}/p \subset S^1$ and consider the central product

$$(2.1) \quad G = G \times_{\langle c \rangle} S^1.$$

Note that $\widetilde{E}_n \cong \widetilde{E}_{n-1} M$, indeed, take $a_{2n} c^{-1/p}$ as a_{2n} , if $a_{2n}^p = c$. Then we have the exact sequence

$$(2.2) \quad 1 \longrightarrow S^1 \longrightarrow \widetilde{G} \longrightarrow V \longrightarrow 1$$

and induced spectral sequence

$$(2.3) \quad E_2^{*,*} \cong H^*(V; H^*(BS^1)) \\ \cong S_{2n} \otimes \wedge_{2n} \otimes \mathbb{Z}/p[u] \implies H^*(\widetilde{G})$$

Then differentials (1.6)-(1.8) also hold but $d_2 = 0$ by the dimensional reason.

Given $H^*(\widetilde{G})$, to see $H^*(G)$ we use the following fibration induced from (2.1)

$$(2.4) \quad S^1 \cong \widetilde{G}/G \longrightarrow BG \longrightarrow B\widetilde{G}.$$

The induced spectral sequence is

$$(2.5) \quad E_2^{*,*} \cong H^*(\widetilde{G}; H^*(S^1)) \\ \cong H^*(\widetilde{G}) \otimes \wedge(z) \implies H^*(G)$$

and $d_2 z = f$ (1.5). Therefore

$$\text{Proposition 2.6.} \quad H^*(G) \cong (\text{Ker}(f) | H^*(G)) \{z\} \oplus H^*(G)/(f).$$

§3. $2p$ -terms for E_n for $n < p$.

In sections 3-5, we consider spectral sequence (2.3) for E_n for $p < n$. Given graded algebra A and $z \in A^{odd}$, we define homology $H(A, z)$ with the differential $d_2(a) = za$. The first non zero differential in (2.3) is $d_3 u = z_n(1)$. Hence

$$(3.1) \quad E_4^{*, 2j} \cong \begin{cases} S_{2n} \otimes \wedge_{2n} / (z_n(1)) & j \equiv 0 \pmod{p} \\ H(S_{2n} \otimes \wedge_{2n}, z_n(1)) & 1 \leq j \leq p-2 \\ \text{Ker } z_n(1) & j = p-1 \end{cases} \\ E_4^{*, 2j+1} = 0.$$

We can prove

$$(3.7) \quad E_{2p}^{* \cdot 2j} \cong \begin{cases} S_{2n} \otimes \Lambda_{2n} / (z_n(1), w_n(1), z_n(2) f^{n-1}) & j=0 \bmod p \\ \mathbb{Z}/p\{f^n\} & 1 \leq j < p-1 \\ 0 & j=p-1. \end{cases}$$

§ 4. E_{2p+2} -term.

The next differential is (1.7)

$$(4.1) \quad d_{2p+1}(u^p) = \partial z_n(1) = z_n(2).$$

Let $E = S_{2n} \otimes \Lambda_{2n} / (z_n(1), w_n(1))$. Then we get from (3.7)

$$(4.2) \quad E_{2p+2}^{* \cdot 2j} = \begin{cases} E/(z_n(2)) & \text{for } j=0 \bmod p \\ H(E/z_n(2) f^{n-1}, z_n(2)) & 0 < j \leq p-2 \\ \text{Ker}(z_n(2) | E/z_n(2) f^{n-1}) & j=p-1 \end{cases}$$

$$E_{2p+2}^{* \cdot 2i+2j} = \mathbb{Z}/p\{f^n\} \quad 0 < i \leq p-2$$

§ 5. Homology of $H(E_i/E_{i+1})$.

Proposition 5.18. $H(E, z_n(2))^{\circ d d} \cong H(E, z_n(2))^{\circ v \circ n} - \mathbb{Z}/p\{f^n\}$

and $H(E, z_n(2))^{\circ d d} \cong S_4\{x_1', \dots, x_{2n}'\} / (y_{ij}x_j', y_i x_k' = y_k x_i')$

where we express $x_i' = x_i f^{n-1}$, $x_i' = y_i f^{n-1}$, $y_i = y_i^p - y_i^{p-1} y_i$.

From (4.2)' we get

Corollary 5.19. $H(E/z_n(2) f^{n-1}, z_n(2))$ is generated by f^{n-1} as

$S_{2n} \otimes \Lambda_{2n}$ -module and

$$H(E/z_n(2) f^{n-1}, z_n(2))^{\circ d d} = H(E, z_n(2))^{\circ d d},$$

$$H(E/z_n(2) f^{n-1}, z_n(2))^{\circ v \circ n} = S_{2n} / (y_i y_{ji}) \{f^{n-1}\} \otimes \mathbb{Z}/p\{f^n\}.$$

§ 6. $E_{2p(p-1)+1}$ -term for \tilde{G} .

Let $y_{ij} = y_i^p - y_i^{p-1} y_j$ and $y_{ij}' = (y_i^{p^2} y_j - y_i y_j^{p^2}) / y_i y_{ji}$

Therefore we can prove

Proposition 6.14. For $n=2$ case

$$E_{2(p-1)p+2}^{\bullet \bullet \bullet 2pj} \cong \begin{cases} S_4 \otimes \Lambda_4 / (z_2(1), z_2(2), w_2(1), w_2(2), (y_{21}' + y_{43}')) \beta(x_1 x_2)) & j=0 \pmod p \\ S_4 \otimes \Lambda_4 / (y_i y_{ji}, y_i x_j - y_j x_i, f, x_h x_k ((h, k) = (1, 2), (3, 4))) & 0 < j < p-1 \pmod p \end{cases}$$

$$E_{2(p-1)p+2}^{\bullet \bullet \bullet 2j} \cong \begin{cases} \mathbb{Z}/p\{x_1 \dots x_4\} & 0 < j < p-1 \pmod p \\ 0 & j=p-1 \pmod p \end{cases}$$

In the next section, we will prove also

Theorem 6.15. $E_{2(p-1)p+2}^{\bullet \bullet \bullet} \cong E_{\infty}^{\bullet \bullet \bullet}$

§ 8. $\text{Ker } f$ in $H^*(E_2)$.

Theorem 8.25. There is an additive isomorphism

$$H^*(E_2) \cong (A/(f) \oplus (\text{Ker}(f)|A)\{z\} \oplus_{1 \leq i \leq p-2} (H^i\{f_i\} \oplus H^i\{f, z\}) \oplus_{1 \leq i \leq p^2-3 \text{ and } i \neq -1, \neq 0 \pmod p \text{ or } i \equiv p(p-1)} (\mathbb{Z}/p\{z_i\} \oplus \mathbb{Z}/p\{z_i\})) \otimes \mathbb{Z}/p[u^{p^2}]$$

where

- (i) $A \cong S_4 \otimes \Lambda_4 / (z_2(1), z_2(2), w_2(1)w_2(2), (y_{21}' + y_{43}')\beta(x_1 x_2), z_2(3))$ with $S_4 \otimes \Lambda_4 = \mathbb{Z}/p[y_1, \dots, y_4] \otimes \Lambda(x_1, \dots, x_4)$, $z_2(1) = \beta f$, $z_2(2) = \mathcal{P}\beta f$, $z_2(3) = \mathcal{P}^2\mathcal{P}\beta f$ and $w_2(1) = \beta z_2(2)$, $w_2(2) = \beta z_2(3)$
- (6.5) $y_{ij}' = y_i^{p(p-1)} + y_i^{(p-1)(p-1)} y_j^{p-1} + \dots + y_j^{p(p-1)}$,
- (ii) $f = \{x_1 x_2 + x_3 x_4\}$,
- (iii) (Proposition 8.2) $\text{Ker}(f)|A$ is generated as an S_4 -module by

$$\begin{aligned} & y_i y_{ji}, y_j x_i - y_i x_j, y_{ji} x_i, \\ & f, x_k x_h ((k, h) \neq (1, 2), \neq (3, 4)), \\ & x_i x_j x_k, x_1 x_2 x_3 x_4, \end{aligned}$$

where $y_{ji} = y_j^p - y_j y_i^{p-1}$,

- (iv) z corresponds non zero element in $H^1(S^1) = E_2^{0,1}$ in (2.4),
- (v) (Proposition 5.18) $H^{\text{odd}} \cong H^{\text{even}} / (\mathbb{Z}/p\{1\})$ and $H^{\text{odd}} \cong S_4\{x_1, \dots, x_4\} / (y_i x_j, y_i x_k = y_k x_i)$,
- (vi) $f_i = \{f u^{p^i}\}$ in the spectral sequence (2.2),
- (vii) $z_i = \{f^2 u^i\} = \{x_1 x_2 x_3 x_4 u^i\}$ in (2.2).

§ 2. Hochschild-Serre spectral sequence.

We consider the spectral sequence with E_2 -term

$$(2.1) \quad E_2^{s,t} = H^s(\oplus^{2n} \mathbb{Z}/p; H^t(BS^1)).$$

In this paper cohomology $H^*(-)$ always means the \mathbb{Z}/p -coefficient $H^*(-; \mathbb{Z}/p)$. Let us write

$$H(+^{2n} \mathbb{Z}/p) = S_{2n} \otimes \wedge_{2n}, \quad H^*(BS^1) \cong \mathbb{Z}/p[u]$$

with $S_{2n} = \mathbb{Z}/p[y_1, \dots, y_{2n}]$, $\wedge_{2n} = \wedge(x_1, \dots, x_{2n})$, $\beta x_i = y_i$.

We assume first non zero differential

$$(2.2) \quad d_3 u = \beta f \text{ with } f = \sum_{i=1}^{2n} x_{2i-1} x_{2i}.$$

Then by Cartan-Serre and Kudo transgression theorems, we know

$$(2.3) \quad d_{2p}^{i-1} (u^{p^{i-1}}) = z(i), \quad d_{2p}^{(p^{i-1}-1)+1} (z(i) \otimes u^{(p-1)p^{i-1}}) = w(i)$$

with $z(i) = p^{p^{i-2}} \dots p^1 \beta f = \sum y_{2j-1}^{p^{i-1}} x_{2j} - y_{2j}^{p^{i-1}} x_{2j-1}$,

$$w(i) = \beta p^{p^{i-1}} z(i) = \sum y_{2j-1}^{p^i} y_{2j} - y_{2j}^{p^i} y_{2j-1}.$$

Let us write $S(i) = S_{2n}/(w(1), \dots, w(i))$. Recall $(w(1), \dots, w(n))$ is regular in S_{2n} [7].

Lemma 2.4. For $i \leq n-1$, we get

(i) 1 is $S(i)$ -free in $E_{2p}^{i+1,*,*}$,

(ii) $z(i+1)$ is $S(i)$ -free in $E_{2p}^{i+1,*,*}$,

(iii) if $x \in E_{2p}^{i+2,*,*}$ is higher $w(i+1)$ -torsion, then x is higher $w(j)$ -torsion for all $j \leq n$ (i.e., $w(j)^s x = 0$ for some s and all $j \leq n$).

(iv) $E_{2p}^{i+2,*,*}$ is higher $w(j)$ -torsion for all $j \leq n$.

For the proof of this lemma, we recall the base wise reduced powers defined by Araki.

Theorem 2.5. (Araki [2]) There are cohomology operations

$$_s P^s : E_r^{a,b} \longrightarrow E_{p(r-2)+2}^{a+(2s-b)(p-1), pb}$$

$$_s \beta P^s : E_r^{a,b} \longrightarrow E_{p(r-2)+2}^{a+(2s-b)(p-1)+1, pb}$$

which satisfy the naturality and Cartan formula.

Proof of Lemma 2.4. We use induction on i . Suppose (i)-(iv) for $i-1$. First we will prove (iv) i.e.,

(1) $H^*(E_{2p}^{i-1, 0}, z(i+1))$ is higher $w(j)$ torsion.

Here $H(A, z)$ means the homology with the differential $da=za$ for $a \in A$. Let us write by $T \subset E_{2p}^{i-1, 0}$ the higher $w(j)$ -torsion parts and $F=E_{2p}^{i-1, 0}/T$. By the inductive assumption, $H(E_{2p}^{i-1, 0}, z(i)) \cong E_{2p}^{i-1, 0} \otimes^{i-1}$ is higher $w(j)$ -torsion. Hence for $2p^{i-1}+2 \leq r \leq 2(p-1)p^{i-1}$, we see $\text{im } d_r \subset T$. Therefore

(2) $E_{r+1}^{*, 0} / (\text{higher } w(j)\text{-torsion}) \cong F$.

Next we consider the Kudo transgression $d_{2p}^{i-1, (p-1)+1}$. Let us write simply $q=2(p-1)p^{i-1}$. Recall that $E_{2p}^{i-1, 0}$ contains $z(i)$ and is a submodule of

$$\text{Ker}(z(i)) \cong H(E_{2p}^{i-1, 0}, z(i)) \oplus \text{Im } z(i).$$

Since $\text{Im } z(i)$ in $E_{q+1}^{*, 0}$ is $S(i-1)$ -free from (ii), if $\text{Ker}(d_{q+1}) \cap \text{Im } z(i) \neq 0$, then it is a contradiction because $E_{q+2}^{*, 0}$ is $w(i)$ -torsion since so is 1. Therefore $\text{Ker}(d_{q+1}) \cap \text{Im } z(i) = 0$. Since $H(E_{2p}^{i-1, 0}, z(i))$ is higher $w(j)$ -torsion, given $a \in E_{q+1}^{*, 0}$ we get $w(i+1)^s a \in \text{Im } z(i)$ for some large s . Hence $E_{q+2}^{*, 0}$ is higher $w(j)$ -torsion. Then we also show, for $2(p-1)p^{i-1}+1 \leq r \leq 2p^i$,

(3) $E_{r+1}^{*, 0} / (\text{higher } w(i)\text{-torsion}) \cong F / (w(i)) / (\text{higher } w(j)\text{-torsion})$.

Let $x \in E_{2p}^{i-1, 0}$ and $x \in \text{Ker } z(i+1)$. From (3) we can write in $E_{2p}^{i-1, 0}$

(4) $z(i+1)x = w(i)a + t$ with t ; higher $w(j)$ -torsion mod $(w(i))$.

Therefore for large s , we have

(5) $z(i+1)w(i+1)^s x = w(i)a'$

We consider Araki's reduced powers

$$\beta \beta^s, \beta \beta^s : E_{2p}^{i-1, 0} \longrightarrow E_{2p}^{i, 0}.$$

Act $\beta \beta^s$ to (5). Since $w(i)=z(i+1)=0$ in $E_{2p}^{i, 0}$ and $\beta \beta^s z(i+1)=w(i+1)$, we get in $E_{2p}^{i, 0}$

(6) $w(i+1)^{s+1}x = w(i+1)\beta a'$.

Multiply $z(i+1)$ to (5), we know $w(i)z(i+1)a'=0$. Act $\beta \beta^{2s}$ to this, and we have

$w(i+1)^2 a' = 0$ in $E_{2p}^{i, 0}$. From (6)

(7) $w(i+1)^{s+2}x = 0$ in $E_{2p}^{i, 0}$.

From (3), this means

$$w(i+1)^{s+2}x = w(i)a''+t' \quad \text{in } E_{2p}^{i-1, 2^{s+2}, 0}.$$

as (4). Multiply $w(i+2)^{s'}$ to this for large s' , we get $w(i+1)^{s''}x = w(i)a''$.

Operate $\mathfrak{p}^{i+1, s''}$ on this. Thus we prove

$$(8) \quad w(i+2)^{s''}x = 0 \quad \text{in } E_{2p}^{i+2, s'', 0}.$$

Continue this argument and we show (1), i.e., (iv). The arguments (7) to (8) implies (iii).

We already know, for $q=2(p-1)p^{i-1}$, $d_{q+1} : S(i-1)\{z(i)\} \cong \text{Ideal}(w(i))$ in $S(i-1)$, by the arguments before (3). Suppose $d_{q+1}(au^{q/2}) = w \neq 0 \pmod{(w(i))}$ in $S(i-1)$ (or $\neq 0 \pmod{(w(i)z(i))}$ in $S(i-1)\{z(i)\}$). Then $w(i+1)^s w \neq 0$ for all s in $S(i-1)/(w(i)) = S(i)$ since $w(i+1)$ is non zero divizer in $S(i)$. On the other hand $H(E_{2p}^{i-1, 1}, z(i))$ is higher $w(j)$ -torsion, we get $w(i+1)^s a \in \text{Im} z(i)$ for large s . This means $w(i+1)^s w = 0 \pmod{(w(i))}$ and this is a contradiction. Hence 1 and $z(i+1)$ are $S(i)$ -free in $E_{q+2^{s+2}, 0}$. From (3), so are in $E_{2p}^{i+1, s+2, 0}$. Therefore we show (i) and (ii). q.e.d.

From (3) in the above proof, we also get;

Corollary 2.6. With modulo higher $w(j)$ -torsion, there is the isomorphism

$$E_{2p}^{i+1, s+2, 0} \cong S_{2n} \otimes \Lambda_{2n} / (z(1), \dots, z(i), w(1), \dots, w(i)).$$

§ 3. Extra special p -groups

let E_n be the extra special p -group of the order $2p^{n+1}$ and the exponent p

$$(3.1) \quad E_n = \langle a_1, \dots, a_{2n}, c \mid a_i^p = c^p = 1, c \in \text{Center} \rangle$$

$$[a_i, a_j] = \begin{cases} c & i=2k-1, j=2k \\ 1 & \text{other } i < j \end{cases}$$

Consider central products $\tilde{E}_n = E_n \times_{\langle c \rangle} S^1$ and $\tilde{E}(s)_n = E_n \times_{\langle c \rangle} \mathbb{Z}/p^s$. Then there are central extensions

$$(3.2) \quad 1 \longrightarrow S^1 \longrightarrow \tilde{E}_n \longrightarrow \oplus^{2n} \mathbb{Z}/p \longrightarrow 1$$

$$(3.3) \quad 1 \longrightarrow \mathbb{Z}/p^s \longrightarrow \tilde{E}(s)_n \longrightarrow \oplus^{2n} \mathbb{Z}/p \longrightarrow 1$$

and induced spectral sequence $E_r^{*,*}$ and $E(s)_r^{*,*}$ from (3.2) and (3.3) respectively. The spectral sequence $E_r^{*,*}$ satisfies (2.2) and hence Lemma 2.4.

Let $H^*(Z/p^*) \cong Z/p[u] \otimes \Lambda(z)$. If $s \geq 2$, then $d_2 z = 0$ by $\beta z = 0$ and the symmetry of $\tilde{E}(s)_n$. Thus

$$(3.4) \quad E(s)_r^{*,*} \cong E_r^{*,*} \otimes \Lambda(z) \quad \text{for } s \geq 2.$$

Therefore (i), (ii) in Lemma 2.4 satisfies for (3.3).

Corollary 3.5. ([7]) In $H^*(\tilde{E}_n)$ or $H^*(\tilde{E}(s)_n)$, $s \geq 2$, the S_{2n} -submodule generated by 1 is $S_{2n}/(w(1), \dots, w(n))$.

Moreover for $n=2$, the spectral sequence $E_r^{*,*}$ is given completely in [8].

§4. Periodic modules with large period.

Let k be an algebraic closure of F_p . Let $\Omega_{G^r}^r(M)$ be the r -th kernel in the minimal resolution of $k(G)$ -module M , i.e., if

$$(4.1) \quad 0 \rightarrow M_r \rightarrow Q_{r-1} \rightarrow \dots \rightarrow Q_0 \rightarrow M \rightarrow 0$$

is exact and if each Q_i is projective, then $M_r \cong \Omega_{G^r}^r(M) \otimes Q$ for some projective module Q . A G -module M is said to be periodic if $\Omega_{G^m}^m(M) \cong M$ for some $m \geq 0$. The smallest of such m is called the period of M .

We denote by $V_G(k)$, the variety defined by commutative ring $H^*(G; k)/\sqrt{0}$. For a G -module M , let $I_G(M)$ be the annihilator in $H^*(G; k)$ of $\text{Ext}_{k(G)}^*(M, M) \cong H^*(G, \text{Hom}_k(M, M))$. Let $V_G(M)$ be the subvariety of $V_G(k)$ associated to $I_G(M)$.

Remark that if V is a closed homogeneous subvariety of $V_G(k)$, then there is a $K(G)$ -module M with $V_G(M) = V$ (Proposition 2.1 (vii) in [3]).

We recall arguments of Andrews and Benson-Carlson [3]. Consider a central extension of a finite group

$$(4.2) \quad 1 \rightarrow Z/p \rightarrow G \rightarrow E \rightarrow 1.$$

Let \bar{Z}/p denote the sum $\sum_{g \in Z/p} g$ as an element of the group ring $k(Z/p)$. Then for $r > 0$, $\bar{Z}/p \Omega_{G^{2r}}^{2r}(k)$ is a $k(G)$ -module with Z/p -acting trivially, so we may regard it as a $k(E)$ -module. We set

$$(4.3) \quad V_r = V_E(\bar{\mathbb{Z}}/p \Omega_G^{2r}(k)) \subset V_E(k).$$

Theorem 4.4. (Andrews) Let M be an indecomposable $k(E)$ -module regard as a $k(G)$ -module by inflation. Then M is a periodic $k(G)$ -module of period dividing $2r$ if and only if $V_E(M) \cap V_r = \{0\}$.

Theorem 4.4 (Benson-Carlson [3]) Let $E_r^{* \cdots}$ be the spectral sequence induced from (4.2). Let $I_p^a \subset H^*(E)$ be the Kernel of the induced map $E_2^{* \cdots} \rightarrow E_{2p^{a+1}}^{* \cdots}$. Then $V_p^a = V_E(I_p^a)$.

Lemma 4.5. ([3] Proposition 2.2.) If M is a periodic $k(G)$ -module, then the period of M divides $2[G;E]$ where E is a maximal elementary abelian p -groups of G .

Theorem 4.6. Let G be the p -group $\tilde{E}(s)_n$, $s \geq 2$. Then there are periodic $K(G)$ -modules of period 2^a for $a \leq n$, and no higher period.

Proof. (See the proof of Corollary 6.2 in [3].) From above lemma, the only possible periods are $2p^a$ for $a \leq n$. By Lemma 2.4 in section 2 and Theorem 4.4, for $a \leq n$ we may find a closed homogeneous subvariety V of $V_E(k)$ with $V \cap V_p^{a-1} \neq \{0\}$ and $V \cap V_p^a = \{0\}$. By the remark after the definition of $V_0(M)$, we may find a $k(E)$ -module M with $V_E(M) = V$. Then by the Andrews theorem $\Omega_E^{2p^{a-1}}(M) \not\cong M$ but $\Omega_E^{2p^a}(M) \cong M$, so M has period exactly $2p^a$. q.e.d.

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